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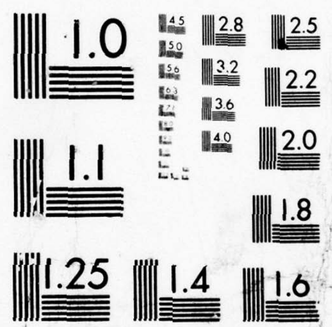

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THE EFFECT OF MASS LOADING ON A STIFFENING RIB

BL Woolley

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NAVAL OCEAN SYSTEMS CENTER
SAN DIEGO, CALIFORNIA 92152

NAVAL OCEAN SYSTEMS CENTER, SAN DIEGO, CA 92152

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RR GAVAZZI, CAPT, USN
Commander

HL BLOOD
Technical Director

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In the scattering of an acoustic wave incident upon a rib-reinforced plate, the effect of the rib can be characterized by a pair of impedances. One of these impedances is associated with the longitudinal vibrations of the rib, and the other impedance is associated with the flexural vibrations of the rib. This report calculates the impedances and the effect of mass loading on these impedances. The calculations are done for a thick or Timoshenko-Mindlin rib which is fluid loaded with the rib immersed either in water or air.

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CONTENTS

SUMMARY

In the scattering of an acoustic wave incident upon a rib-reinforced plate, the effect of the rib can be characterized by a pair of impedances. One of these impedances is associated with the longitudinal vibrations of the rib, and the other impedance is associated with the flexural vibrations of the rib. This report calculates the impedances and the effect of mass loading on these impedances. The calculations are done for a thick or Timoshenko-Mindlin rib which is fluid loaded with the rib immersed either in water or air.

INTRODUCTION

Konovalyuk and Krasil'nikov[1] have attempted to solve the problem of the effect of a reinforcing rib attached to an infinite plane by introducing a pair of localized impedances. At the rib-plate junction, the impedances represent the effect of the spatially distributed rib. Two impedances, Z_f and Z_m, were found to be necessary to represent the different effects of longitudinal and transverse waves in the rib. Konovalyuk and Krasil'nikov treated the case of a thin, fluid-loaded, air-backed plate with a thin rib attached to it.

Graff, et al,[2] extended these calculations to the case of a mass-loaded rib. Their boundary conditions at the rib-plate junction were different than those of Konovalyuk and Krasil'nikov: Graff, et al, treated the case of a continuously welded rib, whereas Konovalyuk and Krasil'nikov treated the case of a spot-welded rib. Graff, et al, also treated only the case of a thin, fluid-loaded, air-backed plate with an attached thin rib.

The purpose of this report is to consider the case of a thick or Timoshenko-Mindlin plate with an attached thick rib. The rib is treated as if it were immersed in air or water, rather than only in air as in references 1 and 2. The rib in this report is also mass loaded. We use the boundary conditions of Graff, et al.

Figure 1 shows the geometry of the problem. A rib of length ℓ is joined perpendicularly to a plate of thickness H. The plate is water loaded and may be backed by water or air. The thickness of the rib is h. The rib is loaded with a distributed, rigid loading at the end farthest from the plate. The load at the end of the rib is characterized by a mass m and a moment of inertia I. The moment of intertia I is calculated about the origin of the Y axis; that origin is at the center point of the thickness of the rib at its point of attachment to the mass loading of the rib. The origin of the Y axis is defined to be centered above the rib at the interface of the loading fluid and the plate.

Mathematically the effect of the rib on the diffraction of the acoustic waves within the fluid loading of the plate is entirely modeled by the two impedances which are functions of the frequency of the incoming wave, the material parameters and geometry of the rib, the mass and moment of inertia of the loading, the conditions which model the junction, and the presence or absence of a fluid in which the rib may be immersed. It is important to note that the lumped nature of the impedances permits the effect of mass loading of the rib on the diffraction of acoustic waves to be considered quite simply. A simple substitution of the

1. Konovalyuk, I. P., and Krasil'nikov, V. N., "The Effect of Stiffening Ribs on Reflection of Plane Sound Waves by a Thin Plate," Problems of Wave Diffraction and Propagation (LGU Collective Publication), No. 4, 1965, p. 149-165.

2. Graff, K. F., Klein, C. A., and Kouyoumjian, R. G., "On the Effect of Mass Loading a Stiffening Rib," Report of the Ohio State University Research Foundation to the Naval Ocean Systems Center, Contract No. N00123-76-C-0728, 10 January 1977.

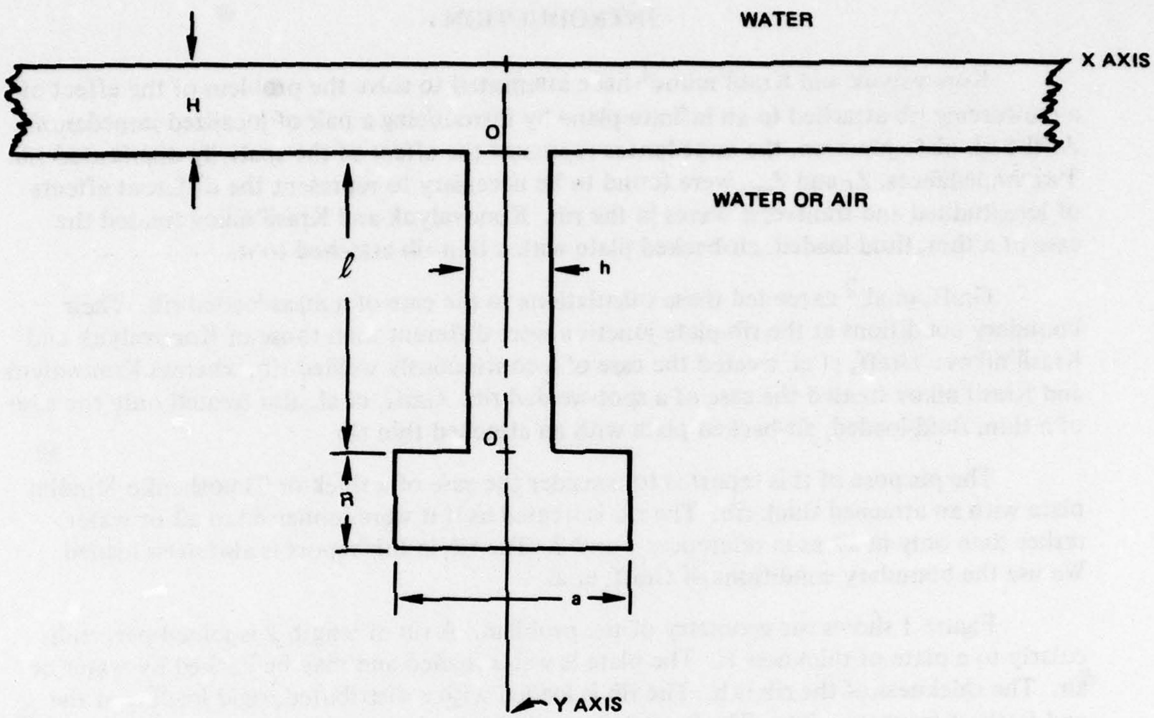

Figure 1. Mass-loaded, rib-stiffened plate

new values for Z_m and Z_f calculated in this report for the values calculated in the papers by Konovalyuk and Krasil'nikov or in those by Woolley[3-7] allows extension of their work to a Timoshenko-Mindlin or thick rib which is mass loaded but immersed in air. Using the simple modifications noted in Woolley's report[7] allows one to extend his work to a Timoshenko-Mindlin or thick rib which is mass loaded with the plate both fluid loaded and fluid backed.

This report first discusses in detail the governing equations of the rib and the boundary conditions at both ends of the rib. The derivations of the longitudinal and flexural impedances are then given in detail. These derivations include demonstrations of how the impedances simplify to the expected values for simple cases. This serves as a check on the validity of the calculations. In the "Discussion and Results" section, the results are conveniently summarized, and the appendix provides details of some of the calculations.

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GOVERNING EQUATIONS

Consider the elements of the stiffening rib as shown in Figure 2. Part A shows the positive sense of the forces for longitudinal vibrations; Part B shows the positive senses of forces and moments for bending vibrations. The longitudinal and transverse displacements, u and v, respectively, are also shown in Figure 2.

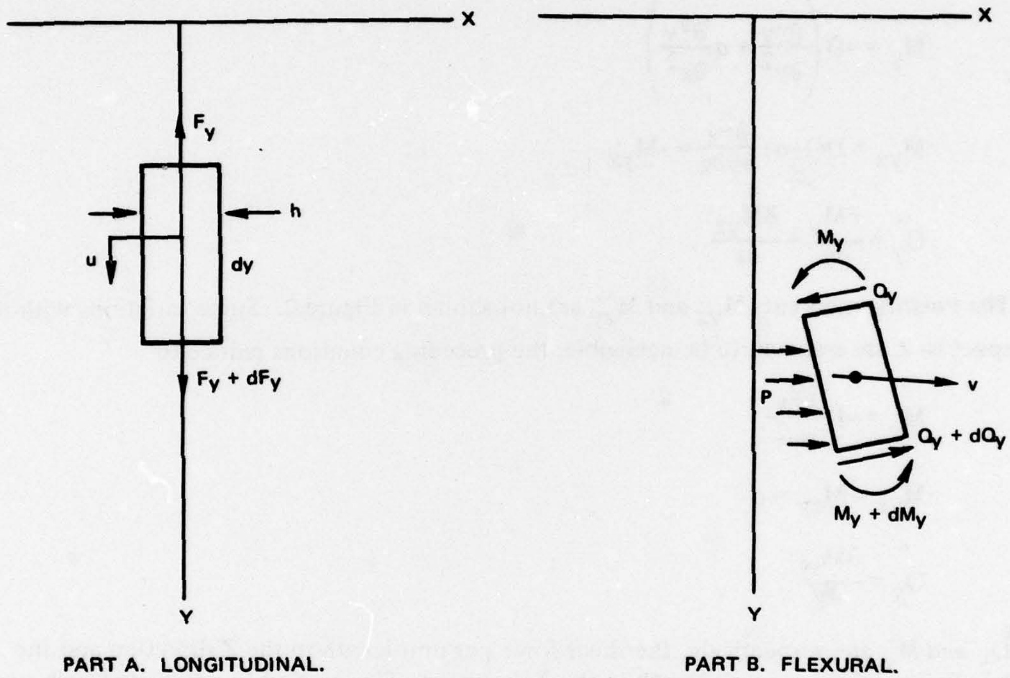

Figure 2. Rib elements under longitudinal and flexural vibrations.

The governing equation for longitudinal vibrations of a rib element is

$$\frac{\partial^2 u}{\partial y^2} = \rho \frac{(1-\sigma^2)}{E} \frac{\partial^2 u}{\partial t^2} , \tag{1}$$

where E, ρ, and σ are Young's modulus, mass per unit volume (per unit length in the Z direction), and Poisson's ratio for the material of the rib, respectively. The force per unit length in the Z direction is

$$F_y = \frac{Eh}{1-\sigma^2} \frac{\partial u}{\partial y} . \tag{2}$$

The governing equation for flexural vibrations in a thick or Timoshenko-Mindlin rib is

$$\left(D\nabla^2 - \frac{\rho h^3}{12}\frac{\partial^2}{\partial t^2}\right)\left(\nabla^2 - \frac{\rho}{\kappa^2 G}\frac{\partial^2}{\partial t^2}\right)v + \rho h\frac{\partial^2 v}{\partial t^2}$$

$$= -\left(1 - \frac{D}{\kappa^2 Gh}\nabla^2 + \frac{\rho h^2}{12\kappa^2 G}\frac{\partial^2}{\partial t^2}\right)p(x,y,t)\Bigg|_{x=o} , \tag{3}$$

where $p(x,y,t)$ is a distributed applied force on the rib, $D = [Eh^3/12(1-\sigma^2)]$ is the cylindrical stiffness of the rib, $\kappa^2 = \pi^2/12$, $G = [E/2(1+\sigma)]$, and $\nabla^2 \equiv (\partial^2/\partial x^2) + (\partial^2/\partial y^2)$. The forces and moments are related to the displacements by

$$\begin{aligned} M_y &= -D\left(\frac{\partial^2 v}{\partial y^2} + \sigma\frac{\partial^2 v}{\partial z^2}\right) \\ M_{yz} &= D(1-\sigma)\frac{\partial^2 v}{\partial y \partial z} = -M_{yz} \\ Q_y &= \frac{\partial M_y}{\partial y} + \frac{\partial M_{yz}}{\partial z} . \end{aligned} \tag{4}$$

The twisting moments M_{yz} and M_{zy} are not shown in Figure 2. Since variations with respect to Z are assumed to be negligible, the preceding equations reduce to

$$\begin{aligned} M_y &= -D\frac{\partial^2 v}{\partial y^2} \\ M_{yz} &= M_{zy} = o \\ Q_y &= \frac{\partial M_{zy}}{\partial y} . \end{aligned} \tag{5}$$

Q_y and M_y are, respectively, the shear force per unit length in the Z direction and the bending moment per unit length in the Z direction. This section has discussed a rib surrounded by air. For the additional equations encountered in the water-immersed case, refer to the appendix.

BOUNDARY CONDITIONS

The flexural and longitudinal impedances are dependent upon the boundary conditions at the rib-plate junction, the type of termination of the rib, and whether the rib is immersed in water or air. We will now present the boundary conditions and discuss the modifications needed in the governing equations for water immersion.

The boundary condition for the longitudinal force is given by

$$\frac{Eh}{1-\sigma^2}\frac{\partial u}{\partial y}\bigg|_{y=o} = -P_o e^{-i\omega t}. \tag{6}$$

We have assumed a harmonic forcing function of $P_o e^{-i\omega t}$. The minus sign on the right side of the equation occurs because the applied force is compressive.

The boundary conditions for bending are

$$-D\frac{\partial^2 v}{\partial y^2}\bigg|_{y=o} = M_o e^{-i\omega t} \tag{7}$$

and

$$v\bigg|_{y=o} = o,$$

that is, we assume a continuous weld so that the shear at the rib-plate junction may be non-zero whereas the transverse displacement is taken to be zero.

The end conditions at the mass-loaded end of the rib can be derived as follows. The mass at the end is taken to act as a simple point mass under longitudinal vibration. This assumes symmetry of the end mass about the Y axis. The end mass is assumed to have sufficient lateral dimensions to require inclusion of rotational inertia effects. This being the case, the equations of motion of the end mass are

$$\begin{aligned} m\frac{\partial^2 u}{\partial t^2}\bigg|_{y=\ell} &= -F_y \\ m\frac{\partial^2 v}{\partial t^2}\bigg|_{y=\ell} &= -Q_y \\ I\frac{\partial^2}{\partial t^2}\left(\frac{\partial v}{\partial y}\right)\bigg|_{y=\ell} &= M_y, \end{aligned} \tag{8}$$

where m is the mass per unit volume per unit length in the Z direction and $I = mr^2$, where r is the radius of gyration.

SUMMARY OF BOUNDARY CONDITIONS

Longitudinal

$$\frac{Eh}{1-\sigma^2}\frac{\partial u}{\partial y}\bigg|_{y=o} = -P_o e^{-i\omega t}$$

$$\frac{Eh}{1-\sigma^2}\frac{\partial u}{\partial y}\bigg|_{y=\ell} = -m\frac{\partial^2 u}{\partial t^2}\bigg|_{y=\ell} \tag{9}$$

Flexural

$$\frac{\partial^2 v}{\partial y^2}\bigg|_{y=o} = -\frac{M_o}{D}e^{-i\omega t}$$

$$v\bigg|_{y=o} = o$$

$$\frac{\partial^2 v}{\partial y^2}\bigg|_{y=\ell} = -\frac{I}{D}\frac{\partial^2}{\partial t^2}\left(\frac{\partial v}{\partial y}\right)\bigg|_{y=\ell} \tag{10}$$

$$\frac{\partial^3 v}{\partial y^3}\bigg|_{y=\ell} = \frac{m}{D}\frac{\partial^2 v}{\partial t^2}\bigg|_{y=\ell}$$

LONGITUDINAL IMPEDANCE SOLUTION

This problem has been correctly solved by K. F. Graff, et al.[2] Their solution is

$$Z_f = \frac{Eh\gamma}{i\omega(1-\sigma^2)} \; \frac{Eh\gamma \sin \gamma\ell + (1-\sigma^2)m\omega^2 \cos \gamma\ell}{Eh\gamma \cos \gamma\ell - (1-\sigma^2)m\omega^2 \sin \gamma\ell} \;, \tag{11}$$

where

$$\gamma = \frac{\omega}{\left(\dfrac{E}{\rho(1-\sigma^2)}\right)^{1/2}} \;. \tag{12}$$

This solution gives physically reasonable results when checked at low frequency and short rib length.

FLEXURAL IMPEDANCE SOLUTION

The governing equations used in the calculation of the flexural impedance of a thick rib are Equations 3 and 4. The appropriate boundary conditions are given by Equation 10. Consider the transverse displacement of the rib to be given by

$$V(y,t) = v(y)e^{-i\omega t} \quad . \tag{13}$$

Now let us first consider the case of an air-backed plate. After we substitute Equation 13 into Equation 3 and set $p(x,y,t) = o$, we have

$$\left[\frac{\partial^4}{\partial y^4} + \rho\omega^2 \left(\frac{1}{\kappa^2 G} + \frac{h^3}{12D}\right) \frac{\partial^2}{\partial y^2} + \frac{\rho^2 h^3 \omega^4}{12\kappa^2 GD} - \frac{\rho h \omega^2}{D}\right] v(y)=o. \tag{14}$$

The solution to this equation has the form

$$v(y) = C_1 \sin \theta y + C_2 \cos \theta y + C_3 \sinh \phi y + C_4 \cosh \phi y, \tag{15}$$

where C_1, C_2, C_3, and C_4 are constants that are determined by the boundary conditions of Equation 10. The details of this solution can be found in the appendix. θ and ϕ are defined in Equation 20.

The flexural impedance is defined as

$$Z_m = \frac{M_y\,(o,t)}{\frac{\partial \Phi}{\partial t}\,(o,t)}, \tag{16}$$

where

$$M_y\,(o,t) = M_o e^{-i\omega t} \tag{17}$$

with $\frac{\partial \Phi}{\partial t}$ (o,t) being the angular velocity of the end of the rib. This is related to the slope of the rib by

$$\frac{\partial \Phi}{\partial t}\,(o,t) = \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial y}\right)\Bigg|_{y=o} . \tag{18}$$

By taking the y derivative of the solution for v, it can be shown (see appendix) that

$$Z_{m \text{ air-immersed}} = \frac{GJ - LE}{- i\omega((E - G)(K + M)\phi + (\theta L - J\phi)(F + H))} , \tag{19}$$

where

$$E = -\theta\left(\theta \sin \theta \ell + \frac{\omega^2 I}{D} \cos \theta \ell\right)$$

$$F = \theta(-\theta \cos \theta \ell + \frac{\omega^2 I}{D} \sin \theta \ell) \frac{1}{D(\theta^2 + \phi^2)}$$

$$G = \phi(\phi \sinh \phi \ell - \frac{\omega^2 I}{D} \cosh \phi \ell)$$

$$H = \phi(\phi \cosh \phi \ell - \frac{\omega^2 I}{D} \sinh \phi \ell) \left(\frac{-1}{D(\theta^2 + \phi^2)} \right)$$

$$J = -\theta^3 \cos \theta \ell + \frac{m\omega^2}{D} \sin \theta \ell \tag{20}$$

$$K = \left(\theta^3 \sin \theta \ell + \frac{m\omega^2}{D} \cos \theta \ell \right) \frac{1}{D(\theta^2 + \phi^2)}$$

$$L = \phi^3 \cosh \phi \ell + \frac{m\omega^2}{D} \sinh \phi \ell$$

$$M = (\phi^3 \sinh \phi \ell + \frac{m\omega^2}{D} \cosh \phi \ell) \left(\frac{-1}{D(\theta^2 + \phi^2)} \right)$$

and

$$\theta = \left(\frac{A + (A^2 - 4B)^{1/2}}{2} \right)^{1/2}$$

$$\phi = \left(\frac{-A + (A^2 - 4B)^{1/2}}{2} \right)^{1/2}$$

with

$$A = \rho\omega^2 \left(\frac{1}{\kappa^2 G} + \frac{h^3}{12D} \right)$$

and

$$B = \frac{\rho h \omega^2}{D} \left(\frac{\rho h^2 \omega^2}{12\kappa^2 G} - 1 \right) .$$

This solution for $Z_{m \text{ air-immersed}}$ reduces to the thin-plate solution given by K. F. Graff, et al. Graff's solution and by extension this solution have been shown to reduce to physically reasonable results in the limit of low frequency and short rib length.

We will consider the calculation of the flexural impedance for the case of a water-backed plate. The governing equations are again Equations 3 and 4. Boundary conditions are again given by Equation 10. However, now $p(x,y,t) \neq 0$ in Equation 3. In addition, $p(x,y,t)$ must satisfy the following two equations:

$$\left. \frac{\partial p}{\partial x} \right|_{x=o} = -\rho_o \frac{\partial^2 v}{\partial t^2}$$

and

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\right) p=o. \tag{21}$$

If we consider the transverse displacement of the rib to be given by

$$V(y,t) = v(y)e^{-i\omega t}, \tag{22}$$

then substituting this into Equation 3 gives

$$\left[\frac{\partial^4}{\partial y^4}+\rho\omega^2\left(\frac{1}{\kappa^2 G}+\frac{h^3}{12D}\right)\frac{\partial^2}{\partial y^2}+\frac{\rho^2h^3\omega^4}{12\kappa^2GD}-\frac{\rho h\omega^2}{D}\right] v(y)$$
$$= -\left(1-\frac{D}{\kappa^2 Gh}\frac{\partial^2}{\partial y^2}+\frac{\rho h^2}{12\kappa^2 G}\frac{\partial^2}{\partial t^2}\right)\frac{p(x,y,t)}{D}\Bigg|_{x=o} . \tag{23}$$

We assume that $p(x,y,t) = P(x,y)e^{-i\omega t}$ and we know that the general solution for $V(y)$* is

$$V(y) = C_0 \sin k_1y + C_1 \cos k_1y + C_2 \sinh k_2y + C_3 \cosh k_2y. \tag{24}$$

C_0, C_1, C_2, and C_3 are constants that are determined by the boundary conditions of Equation 10.* k_1 and k_2 are defined as follows:

$$k_1 = \left(\frac{+A+(A^2-4B)^{1/2}}{2}\right)^{1/2} \text{ and } k_2=\left(\frac{-A'+(A'^2-4B')^{1/2}}{2}\right)^{1/2}, \tag{25}$$

where

$$A = \omega^2\left(\frac{\rho}{\kappa^2 G}+\frac{h^3\rho}{12D}-\frac{2\rho_o}{\kappa^2 Gh\gamma_1}\right)$$

$$A' = \omega^2\left(\frac{\rho}{\kappa^2 G}+\frac{h^3\rho}{12D}-\frac{2\rho_o}{\kappa^2 Gh\gamma_2}\right)$$

$$B = \frac{\rho^2h^3\omega^4}{12\kappa^2GD}-\frac{\rho h\omega^2}{D}+\frac{2\rho_o\omega^2}{D}\left(\frac{1}{\gamma_1}-\frac{\rho h^2\omega^2}{12\kappa^2G\gamma_1}\right)$$

and

$$B' = \frac{\rho^2h^3\omega^4}{12\kappa^2GD}-\frac{\rho h\omega^2}{D}+\frac{2\rho_o\omega^2}{D}\left(\frac{1}{\gamma_2}-\frac{\rho h^2\omega^2}{12\kappa^2G\gamma_2}\right)$$

with $\gamma_1^2 = k^2-k_1^2$ and $\gamma_2^2 = k^2+k_2^2$. For a thin plate, $A = A' = 0$, $B = -(\rho h\omega^2/D) + (\rho_o\omega^2/D\gamma_1)$, and $B' = -(\rho h\omega^2/D)+(\rho_o\omega^2/D\gamma_2)$. k_1 and k_2 have to be defined as they have been in order for Equation 23 to be solved by a $p(x,y,z)$ which has the form

*See appendix for detailed calculations.

$$p(x,y,t) = \rho_o \omega^2 e^{-i\omega t} \left[\left(\frac{C_o \sin k_1 y + C_1 \cos k_1 y}{\gamma_1} \right) e^{i\gamma_1 x} + \left(\frac{C_2 \sinh k_2 y + C_3 \cosh k_2 y}{\gamma_2} \right) e^{i\gamma_2 x} \right]. \tag{26}$$

This form of p(x,y,t) was chosen because it, along with v(y), solves Equation 21.

The boundary conditions of Equation 10 are now used to obtain C_0, C_1, C_2, and C_3:

$$\begin{aligned} C_0 &= \frac{-(E+F)\,H + B\,(J+L)}{AH-GB} \\ C_1 &= \frac{M_o}{D(k_1^2 + k_2^2)} \\ C_2 &= \frac{-G(E+F) + A\,(J+L)}{BG-AH} \end{aligned} \tag{27}$$

and

$$C_3 = \frac{-M_o}{D(k_1^2 + k_2^2)} ,$$

where the coefficients are defined as follows:

$$\begin{aligned} A &= -k_1^2 \sin k_1 \ell - \frac{I\omega^2}{D} k_1 \cos k_1 \ell \\ B &= k_2^2 \sinh k_2 \ell - \frac{I\omega^2}{D} k_2 \cosh k_2 \ell \\ E &= \frac{M_o}{D(k_1^2 + k_2^2)} \left(- k_1^2 \cos k_1 \ell + \frac{I\omega^2}{D} k_1 \sin k_1 \ell \right) \\ F &= \frac{M_o}{D(k_1^2 + k_2^2)} \left(- k_2^2 \cosh k_2 \ell + \frac{I\omega^2}{D} k_2 \sinh k_2 \ell \right) \\ G &= - k_1^3 \cos k_1 \ell + \frac{m\omega^2}{D} \sin k_1 \ell \\ H &= k_2^3 \cosh k_2 \ell + \frac{m\omega^2}{D} \sinh k_2 \ell \\ J &= \frac{M_o}{D(k_1^2 + k_2^2)} \left(k_1^3 \sin k_1 \ell + \frac{m\omega^2}{D} \cos k_1 \ell \right) \end{aligned} \tag{28}$$

and

$$L = \frac{M_o}{D(k_1^2 + k_2^2)} \left(-k_2^3 \sinh k_2 \ell - \frac{m\omega^2}{D} \cosh k_2 \ell \right) .$$

The flexural impedance is defined as

$$Z_m = \frac{M_y(o,t)}{\frac{\partial \Phi}{\partial t}(o,t)} , \tag{29}$$

where

$$M_y(o,t) = M_o e^{-i\omega t} \tag{30}$$

with $\frac{\partial \Phi}{\partial t}(o,t)$ being the angular velocity of the end of the rib. This is related to the slope of the rib by

$$\frac{\partial \Phi}{\partial t}(o,t) = \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial y}\right) \Bigg|_{\, y = o} . \tag{31}$$

By taking the y derivative of the solution for v, it can be shown (see appendix) that

$$Z_{m \text{ water-immersed}} = \frac{AH - GB}{-i\omega\left((E' + F')(-k_1 H + k_2 G) + (J' + L')(k_1 B - k_2 A)\right)} , \tag{32}$$

where

$$\begin{aligned}
E' &= \frac{1}{D(k_1^2 + k_2^2)} \left(-k_1^2 \cos k_1 \ell + \frac{I\omega^2}{D} k_1 \sin k_1 \ell \right) \\
F' &= \frac{1}{D(k_1^2 + k_2^2)} \left(-k_2^2 \cosh k_2 \ell + \frac{I\omega^2}{D} k_2 \sinh k_2 \ell \right) \\
J' &= \frac{1}{D(k_1^2 + k_2^2)} \left(k_1^3 \sin k_1 \ell + \frac{m\omega^2}{D} \cos k_1 \ell \right) \\
L' &= \frac{1}{D(k_1^2 + k_2^2)} \left(-k_2^2 \sinh k_2 \ell - \frac{m\omega^2}{D} \cosh k_2 \ell \right) .
\end{aligned} \tag{33}$$

DISCUSSION AND RESULTS

The longitudinal impedance for a mass-loaded and air-immersed thick rib of thickness h and length ℓ is

$$Z_F = \frac{Eh\gamma}{i\omega\,(1-\sigma^2)} \quad \frac{Eh\gamma \sin\gamma\ell + (1-\sigma^2)\, m\omega^2 \cos\gamma\ell}{Eh\gamma \cos\gamma\ell - (1-\sigma^2)\, m\omega^2 \sin\gamma\ell}, \tag{34}$$

where

$$\gamma = \frac{\omega}{\left(\dfrac{E}{\rho(1-\sigma)}\right)^{1/2}} .$$

When the rib is water immersed, the right side of the second equation of the boundary conditions (Equation 9) has the added term $+\, b\dfrac{\partial u}{\partial t}\Big|_{y=\ell}$, where b is a damping coefficient. This changes the Equation 34 for Z_F by replacing the three terms of $Eh\gamma$ with $Eh\gamma - ib\omega(1-\sigma^2)$.

The flexural impedance for a mass-loaded and air-immersed thick rib of thickness h and length ℓ is

$$Z_{M\ \text{air-immersed}} = \frac{GJ - LE}{-\,i\omega((E-G)\,(k+M)\,\phi + (\theta L - J\phi)\,(F+H))} \ , \tag{35}$$

where

$$\begin{aligned}
E &= -\theta\,(\theta \sin\theta\ell + \frac{\omega^2 I}{D}\cos\theta\ell) \\
F &= \theta\,(-\theta \cos\theta\ell + \frac{\omega^2 I}{D}\sin\theta\ell)\,\frac{1}{D(\theta^2+\phi^2)} \\
G &= \phi\,(\phi \sinh\phi\ell - \frac{\omega^2 I}{D}\cosh\phi\ell) \\
H &= \phi\,(\phi \cosh\phi\ell - \frac{\omega^2 I}{D}\sinh\phi\ell)\left(\frac{-1}{D(\theta^2+\phi^2)}\right) \\
J &= -\theta^3 \cos\theta\ell + \frac{m\omega^2}{D}\sin\theta\ell \\
K &= (\theta^3 \sin\theta\ell + \frac{m\omega^2}{D}\cos\theta\ell)\,\frac{1}{D(\theta^2+\phi^2)}
\end{aligned} \tag{36}$$

$$L = \phi^3 \cosh \phi\ell + \frac{m\omega^2}{D} \sinh \phi\ell$$

$$M = \left(\phi^3 \sinh \phi\ell + \frac{m\omega^2}{D} \cosh \phi\ell\right) \left(\frac{-1}{D(\theta^2 + \phi^2)}\right)$$

and

$$\theta = \left(\frac{A + (A^2 - 4B)^{1/2}}{2}\right)^{1/2}$$

$$\phi = \left(\frac{-A + (A^2 - 4B)^{1/2}}{2}\right)^{1/2}$$

with

$$A = \rho\omega^2 \left(\frac{1}{\kappa^2 G} + \frac{h^3}{12D}\right) \quad \text{and} \quad B = \frac{\rho h \omega^2}{D} \left(\frac{\rho h^2 \omega^2}{12\kappa^2 G} - 1\right) .$$

The flexural impedance for a mass-loaded and water-immersed thick rib of thickness h and length ℓ is

$$Z_{m \text{ water-immersed}} = \frac{AH - GB}{-i\omega \ \left((E' + F')(-k_1 H + k_2 G) + (J' + L')(k_1 B - k_2 A)\right)} , \quad (37)$$

where

$$A = -k_1^2 \sin k_1\ell - \frac{I\omega^2}{D} k_1 \cos k_1\ell$$

$$B = k_2^2 \sinh k_2\ell - \frac{I\omega^2}{D} k_2 \cosh k_2\ell$$

$$E' = \frac{1}{D(k_1^2 + k_2^2)} \left(-k_1^2 \cos k_1\ell + \frac{I\omega^2}{D} k_1 \sin k_1\ell\right)$$

$$F' = \frac{1}{D(k_1^2 + k_2^2)} \left(-k_2^2 \cosh k_2\ell + \frac{I\omega^2}{D} k_2 \sinh k_2\ell\right) \quad (38)$$

$$G = -k_1^3 \cos k_1\ell + \frac{m\omega^2}{D} \sin k_1\ell$$

$$H = k_2^3 \cosh k_2\ell + \frac{m\omega^2}{D} \sinh k_2\ell$$

$$J' = \frac{1}{D(k_1^2 + k_2^2)} \left(k_1^3 \sin k_1\ell + \frac{m\omega^2}{D} \cos k_1\ell\right)$$

$$L' = \frac{1}{D(k_1^2 + k_2^2)} \left(- k_2^3 \sinh k_2 \ell - \frac{m\omega^2}{D} \cosh k_2\right) \text{ and}$$

k_1 and k_2 can be found from the following two equations:

$$k_1 = \left(\frac{+ A + (A^2 - 4B)^{1/2}}{2}\right)^{1/2}$$

$$k_2 = \left(\frac{-A' + (A'^2 - 4B')^{1/2}}{2}\right)^{1/2},$$

where

$$A = \omega^2\left(\frac{\rho}{\kappa^2 G} + \frac{h^3 \rho}{12D} - \frac{2\rho_o}{\kappa^2 Gh(k^2 - k_1^2)^{1/2}}\right)$$

$$B = \frac{\rho^2 h^3 \omega^4}{12\kappa^2 GD} - \frac{\rho h \omega^2}{D} + \frac{2\rho_o \omega^2}{D}\left(1 - \frac{\rho h^2 \omega^2}{12\kappa^2 G}\right) \frac{1}{(k^2 - k_1^2)^{1/2}}$$

$$A' = \omega^2\left(\frac{\rho}{\kappa^2 G} + \frac{h^3 \rho}{12D} - \frac{2\rho_o}{\kappa^2 Gh\,(k^2 + k_2^2)^{1/2}}\right)$$

and

$$B' = -\frac{\rho h \omega^2}{D} + \frac{\rho_o \omega^2}{D(k^2 + k_2^2)^{1/2}}.$$

When the effects of fluid damping are taken into account, the last two of the boundary conditions of Equation 10 are modified to become

$$\left. - b' \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial y}\right)\right|_{y=\ell} - D \left.\frac{\partial^2 v}{\partial y^2}\right|_{y=\ell} = I \left.\frac{\partial^2}{\partial t^2}\left(\frac{\partial v}{\partial y}\right)\right|_{y=\ell}$$

$$\left. - b' \frac{\partial v}{\partial t}\right|_{y=\ell} + D \left.\frac{\partial^3 v}{\partial y^3}\right|_{y=\ell} = m \left.\frac{\partial^2 v}{\partial t^2}\right|_{y=\ell}, \qquad (39)$$

where b' is a damping coefficient. This modifies Equation 37 for Z_m water-immersed by replacing $\frac{\omega^2 I}{D}$ with $\frac{\omega^2 I - i\omega b'}{D}$ and $\frac{m\omega^2}{D}$ with $\frac{m\omega^2 + i\omega b'}{D}$ wherever they occur in Equation 38.

Thus this report provides the capability to handle acoustic scattering from plates with mass-loaded ribs, if used in conjunction with previous work. The mass loading must be symmetrical to be handled by the theory in this report.

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7. Available to qualified requestors.

APPENDIX. DETAILED CALCULATIONS FOR FLEXURAL IMPEDANCES

This appendix gives the details of the solution for the flexural impedances when the rib is immersed in air and when the rib is immersed in water.

The flexural wave equation is

$$\left(D\nabla^2 - \frac{\rho h^3}{12}\frac{\partial^2}{\partial t^2}\right)\left(\nabla^2 - \frac{\rho}{\kappa^2 G}\frac{\partial^2}{\partial t^2}\right) v + \rho h \frac{\partial^2 v}{\partial t^2}$$

$$= -\left(1 - \frac{D}{\kappa^2 G h}\nabla^2 + \frac{\rho h^2}{12\kappa^2 G}\frac{\partial^2}{\partial t^2}\right) p(x,y,t)\Big|_{x=o} \quad . \qquad \text{(A-1)}$$

Assume that v is a function of y with a harmonic time dependence

$$v = v(y)e^{-i\omega t}. \qquad \text{(A-2)}$$

The flexural wave equation then becomes

$$\left[\frac{\partial^4}{\partial y^4} + \rho\omega^2\left(\frac{1}{\kappa^2 G} + \frac{h^3}{12D}\right)\frac{\partial^2}{\partial y^2} + \frac{\rho^2 h^3 \omega^4}{12\kappa^2 GD} - \frac{\rho h \omega^2}{D}\right] v(y) = o \qquad \text{(A-3)}$$

for an air-immersed rib, and

$$\left[\frac{\partial^4}{\partial y^4} + \rho\omega^2\left(\frac{1}{\kappa^2 G} + \frac{h^3}{12D}\right)\frac{\partial^2}{\partial y^2} + \frac{\rho^2 h^3 \omega^4}{12\kappa^2 GD} - \frac{\rho h \omega^2}{D}\right] v(y)$$

$$= -\left(1 - \frac{D}{\kappa^2 G h}\nabla^2 + \frac{\rho h^2}{12\kappa^2 G}\frac{\partial^2}{\partial t^2}\right) p(x,y,t)\Big|_{x=o} \qquad \text{(A-4)}$$

for a water-immersed rib. We will now discuss these cases.

For an air-immersed rib we can write the flexural wave equation as

$$\left(\frac{\partial^4}{\partial y^4} + A\frac{\partial^2}{\partial y^2} + B\right) v(y) - C\, v(y) = o, \qquad \text{(A-5)}$$

where

$$A = \rho\omega^2\left(\frac{1}{\kappa^2 G} + \frac{h^3}{12D}\right)$$

$$B = \frac{\rho h^3 \omega^4}{12\kappa^2 GD}$$

$$C = \frac{\rho h \omega^2}{D} \quad .$$

For a thin plate A = o and B = o. The boundary conditions are

$$\left.\frac{\partial^2 v}{\partial y^2}\right|_{y=o} = -\frac{M_o}{D}$$

$$\left. v \right|_{y=o} = o$$

$$\left.\frac{a^2 v}{a y^2}\right|_{y=\ell} = \left.\frac{\omega^2 I}{D}\frac{\partial v}{\partial y}\right|_{y=\ell} \tag{A-6}$$

$$\left.\frac{\partial^3 v}{\partial y^3}\right|_{y=\ell} = -\left.\frac{m\omega^2}{D} v\right|_{y=\ell}$$

We can assume a general solution of the form

$$v(y) = C_1 \sin \theta y + C_2 \cos \theta y + C_3 \sinh \phi y + C_4 \cosh \phi y \tag{A-7}$$

with the following derivatives:

$$\frac{\partial v}{\partial y} = \theta C_1 \cos \theta y - \theta C_2 \sin \theta y + \phi C_3 \cosh \phi h + \phi C_4 \sinh \phi y$$

$$\frac{\partial^2 v}{\partial y^2} = -\theta^2 C_1 \sin \theta y - \theta^2 C_2 \cos \theta y + \phi^2 C_3 \sinh \phi y + \phi^2 C_4 \cosh \phi y \tag{A-8}$$

$$\frac{\partial^3 v}{\partial y^3} = -\theta^3 C_1 \cos \theta y + \theta^3 C_2 \sin \theta y + \phi^3 C_3 \cosh \phi y + \phi^3 C_4 \sinh \phi y$$

$$\frac{\partial^4 v}{\partial y^4} = \theta^4 C_1 \sin \theta y + \theta^4 C_2 \cos \theta y + \phi^4 C_3 \sinh \phi y + \phi^4 C_4 \cosh \phi y.$$

Substituting these expressions into Equation A-5 gives

$$\theta = \pm\sqrt{\frac{A \pm \sqrt{A^2 - 4D}}{2}} = \sqrt{\frac{A + \sqrt{A^2 - 4D}}{2}}$$

and

$$\phi = \pm\sqrt{\frac{-A \pm \sqrt{A^2 - 4D}}{2}} = \sqrt{\frac{-A + \sqrt{A^2 - 4D}}{2}} \tag{A-9}$$

with D = B – C and A as previously defined. The choice of signs has been made so that

$$\theta = \left(\frac{\omega^2 \rho h}{D}\right)^{1/4} \quad \text{and} \quad \phi = \left(\frac{\omega^2 \rho h}{D}\right)^{1/4}, \tag{A-10}$$

when we take the thin-plate case, i.e., $A \equiv 0$, instead of the thick-plate case.

Substituting these expressions (Equation A-8) into the boundary conditions gives the following equations:

$$
\begin{aligned}
& -\theta^2 C_2 + \phi^2 C_4 = -\frac{M_o}{D} \\
& C_2 + C_4 = o \qquad \text{(A-11)} \\
& -\theta^2 C_1 \sin\theta\ell - \theta^2 C_2 \cos\theta\ell + \phi^2 C_3 \sinh\phi\ell + \phi^2 C_4 \cosh\phi\ell \\
& \quad = \frac{\omega^2 I}{D}\left(\theta C_1 \cos\theta\ell - \theta C_2 \sin\theta\ell + \phi C_3 \cosh\phi\ell + \phi C_4 \sinh\phi\ell\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& -\theta^3 C_1 \cos\theta\ell + \theta^3 C_2 \sin\theta\ell + \phi^3 C_3 \cosh\phi\ell + \phi^3 C_4 \sinh\phi\ell \\
& \quad = -\frac{m\omega^2}{D}\left(C_1 \sin\theta\ell + C_2 \cos\theta\ell + C_3 \sinh\phi\ell + C_4 \cosh\phi\ell\right) .
\end{aligned}
$$

From the first and second of the above equations, we obtain

$$
C_2 = -C_4 = \frac{M_o}{D(\theta^2 + \phi^2)} . \qquad \text{(A-12)}
$$

Using these results in the last two expressions of Equation A-11 gives

$$
C_1 = -\left(\frac{E(K+M) - J(F+H)}{GJ - LE}\right)\frac{G}{E} - \left(\frac{F+H}{E}\right) \qquad \text{(A-13)}
$$

and

$$
C_3 = \frac{E(K+M) - J(F+H)}{GJ - LE}
$$

with

$$
\begin{aligned}
E &= -\theta\left(\theta \sin\theta\ell + \frac{\omega^2 I}{D}\cos\theta\ell\right) \\
F &= \theta\left(-\theta\cos\theta\ell + \frac{\omega^2 I}{D}\sin\theta\ell\right)\frac{M_o}{D(\theta^2+\phi^2)} \\
G &= \phi\left(\phi\sinh\phi\ell - \frac{\omega^2 I}{D}\cosh\phi\ell\right) \\
H &= \phi\left(\phi\cosh\phi\ell - \frac{\omega^2 I}{D}\sinh\phi\ell\right)\left(\frac{-M_o}{D(\theta^2+\phi^2)}\right) \qquad \text{(A-14)} \\
J &= -\theta^3\cos\theta\ell + \frac{m\omega^2}{D}\sin\theta\ell
\end{aligned}
$$

$$K = \left(\theta^3 \sin \theta\ell + \frac{m\omega^2}{D} \cos \theta\ell\right) \frac{M_o}{D(\theta^2 + \phi^2)}$$

$$L = \phi^3 \cosh \phi\ell + \frac{m\omega^2}{D} \sinh \phi\ell$$

and

$$M = \left(\phi^3 \sinh \phi\ell + \frac{m\omega^2}{D} \cosh \phi\ell\right)\left(\frac{-M_o}{D(\theta^2 + \phi^2)}\right).$$

Now the flexural impedance is

$$Z_m = \frac{M_o e^{-i\omega t}}{\frac{\partial \phi}{\partial t}(o,t)}$$

with

$$\frac{\partial \phi}{\partial t}(o,t) = \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial y}\right)\Bigg|_{y=o}. \qquad \text{(A-15)}$$

Using the solution for v gives

$$\frac{\partial \phi}{\partial t}(o,t) = -i\omega\,(\theta C_1 + \phi C_3)e^{-i\omega t}.$$

Thus

$$Z_{m\ \text{air-immersed}} = \frac{M_o e^{-i\omega t}}{-i\omega(\theta C_1 + \phi C_3)e^{-i\omega t}} = \frac{GJ - LE}{-i\omega((E - G)(K + M)\phi + (\theta L - J\phi)(F + H))}, \qquad \text{(A-16)}$$

where E, G, J, and L are as formerly and

$$F = \theta\left(-\theta \cos \theta\ell + \frac{\omega^2 I}{D} \sin \theta\ell\right) \frac{1}{D(\theta^2 + \phi^2)}$$

$$H = \phi\left(\phi \cosh \phi\ell - \frac{\omega^2 I}{D} \sinh \phi\ell\right)\left(\frac{-1}{D(\theta^2 + \phi^2)}\right)$$

$$K = \left(\theta^3 \sin \theta\ell + \frac{m\omega^2}{D} \cos \theta\ell\right) \frac{1}{D(\theta^2 + \phi^2)}$$

and

$$M = \left(\phi^3 \sinh \phi\ell + \frac{m\omega^2}{D} \cosh \phi\ell\right)\left(\frac{-1}{D(\theta^2 + \phi^2)}\right).$$

For a water-immersed rib the flexural wave equation is

$$\left[\frac{\partial^4}{\partial y^4}+\rho\omega^2\left(\frac{1}{\kappa^2 G}+\frac{h^3}{12D}\right)\frac{\partial^2}{\partial y^2}+\frac{\rho^2 h^3\omega^4}{12\kappa^2 GD}-\frac{\rho h\omega^2}{D}\right]v(y)$$

$$=-\left(1-\frac{D}{\kappa^2 Gh}\frac{\partial^2}{\partial y^2}+\frac{\rho h^2}{12\kappa^2 G}\frac{\partial^2}{\partial t^2}\right)\frac{p(x,y,t)}{D}\Bigg|_{x=o} . \tag{A-17}$$

p(x,y,t) must solve the following two equations:

$$\frac{\partial p}{\partial x}\Bigg|_{x=o}=-\rho_o\frac{\partial^2 v}{\partial t^2} \tag{A-18}$$

and

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\right)p=o. \tag{A-19}$$

We let

$$p(x,y,t)=P(x,y)e^{-i\omega t}. \tag{A-20}$$

We now assume a general solution for V(y) in the form

$$V(y)=C_o\sin k_1y+C_1\cos k_1y+C_2\sinh k_2y+C_3\cosh k_2y$$
$$=V_1(k_1,y)+V_2(k_2,y). \tag{A-21}$$

From Equations A-19 and A-20 we find

$$p(x,y,t)=\rho_o\omega^2e^{-i\omega t}\left[\left(\frac{C_o\sin k_1y+C_1\cos k_1y}{\gamma_1}\right)e^{i\gamma_1x}\right.$$
$$\left.+\left(\frac{C_2\sinh k_2y+C_3\cosh k_2y}{\gamma_2}\right)e^{i\gamma_2x}\right], \tag{A-22}$$

where $\gamma_1^2=k^2-k_1^2$ and $\gamma_2^2=k^2+k_2^2$. Equations A-21 and A-22 are substituted into Equation A-17 to obtain

$$\left(\frac{\partial^4}{\partial y^4}+A\frac{\partial^2}{\partial y^2}+B\right)V_1+\left(\frac{\partial^4}{\partial y^4}+A'\frac{\partial^2}{\partial y^2}+B'\right)V_2=0, \tag{A-23}$$

where

$$A=\omega^2\left(\frac{\rho}{\kappa^2G}+\frac{h^3\rho}{12D}-\frac{\alpha\rho_o i}{\kappa^2Gh\gamma_1}\right)$$

$$A' = \omega^2 \left(\frac{\rho}{\kappa^2 G} + \frac{h^3 \rho}{12D} - \frac{\alpha \rho_o i}{\kappa^2 G h \gamma_2} \right)$$

$$B = \frac{\rho^2 h^3 \omega^4}{12\kappa^2 GD} - \frac{\rho h \omega^2}{D} + \frac{\alpha \rho_o \omega^2}{D} \left(\frac{1}{\gamma_1} - \frac{\rho h^2 \omega^2}{12\kappa^2 G \gamma_1} \right)$$

and

$$B' = \frac{\rho^2 h^3 \omega^4}{12\kappa^2 GD} - \frac{\rho h \omega^2}{D} + \frac{\alpha \rho_o \omega^2}{D} \left(\frac{1}{\gamma_2} - \frac{\rho h^2 \omega^2}{12\kappa^2 G \gamma_2} \right).$$

For a thin plate $A = A' = 0$, $B = -(\rho h \omega^2/D) + (\rho_o \omega^2/D\gamma_1)$, and $B' = -(\rho h \omega^2/D) + (\rho_o \omega^2/D\gamma_2)$. $\alpha = 2$ for water on both sides of the rib, i.e., for the case under consideration, and $\alpha = 1$ for water on one side of the rib. Equation A-23 only has a solution if

$$k_1 = \left(\frac{+A + (A^2 - 4B)^{1/2}}{2} \right)^{1/2} \text{ and } k_2 = \left(\frac{-A' + (A'^2 - 4B')^{1/2}}{2} \right)^{1/2} \quad . \tag{A-24}$$

Using Equation A-24 and the relationships

$$\gamma_1^2 = k^2 - k_1^2 \text{ and } \gamma_2^2 = k^2 - k_2^2,$$

a fifth-order equation for γ_1 or γ_2 can be obtained. The solution of this equation can then be used in conjunction with Equation A-24 to find k_1 and k_2. To solve the problem completely we only need to find the coefficients C_0, C_1, C_2, and C_3. This is done by substituting Equation A-21 and its derivatives into the boundary conditions given by Equation A-6.

We will first present the derivatives of V(y) with respect to y:

$$\frac{\partial V}{\partial y} = k_1 C_0 \cos k_1 y - k_1 C_1 \sin k_1 y + k_2 C_2 \cosh k_2 y + k_2 C_3 \sinh k_2 y$$

$$\frac{\partial^2 V}{\partial y^2} = -k_1^2 C_0 \sin k_1 y - k_1^2 C_1 \cos k_1 y + k_2^2 C_2 \sinh k_2 y + k_2^2 C_3 \cosh k_2 y$$

$$\frac{\partial^3 V}{\partial y^3} = -k_1^3 C_0 \cos k_1 y + k_1^3 C_1 \sin k_1 y + k_2^3 C_2 \cosh k_2 y + k_2^3 C_3 \sinh k_2 y \tag{A-25}$$

$$\frac{\partial^4 V}{\partial y^4} = k_1^4 C_0 \sin k_1 y + k_1^4 C_1 \cos k_1 y + k_2^4 C_2 \sinh k_2 y + k_2^4 C_3 \cosh k_2 y.$$

Substituting the expressions in Equation A-25 into the boundary conditions (Equation A-6) provides the following four equations:

$$C_1 + C_3 = 0$$

$$-D(-k_1^2 C_1 + k_2^2 + C_3) = M_o$$

$$-k_1^2 C_0 \sin k_1\ell - k_1^2 C_1 \cos k_1\ell + k_2^2 C_2 \sinh k_2\ell + k_2^2 C_3 \cosh k_2\ell \tag{A-26}$$

$$-\frac{I\omega^2}{D}\left(k_1C_0 \cos k_1\ell - k_1C_1 \sin k_1\ell + k_2C_2 \cosh k_2\ell + k_2C_3 \sinh k_2\ell\right) = 0$$

and

$$-k_1^3 C_0 \cos k_1\ell + k_1^3 C_1 \sin k_1\ell + k_2^3 C_2 \cosh k_2\ell + k_2^3 C_3 \sinh k_2\ell$$

$$+\frac{m\omega^2}{D}\left(C_0 \sin k_1\ell + C_1 \cos k_1\ell + C_2 \sinh k_2\ell + C_3 \cosh k_2\ell\right) = 0$$

From the first and second expressions in Equation A-26, we obtain

$$C_1 = \frac{M_o}{D(k_1^2 + k_2^2)} \text{ and } C_3 = -\frac{M_o}{D(k_1^2 + k_2^2)} \ . \tag{A-27}$$

Using these results in the last two expressions of Equation A-26 provides

$$C_0 = \frac{-(E + F)H + B(J + L)}{AH - GB} \tag{A-28}$$

and

$$C_2 = \frac{-G(E + F) + A(J + L)}{BG - AH} \tag{A-28}$$

with

$$A = -k_1^2 \sin k_1\ell - \frac{I\omega^2}{D} k_1 \cos k_1\ell$$

$$B = k_2^2 \sinh k_2\ell - \frac{I\omega^2}{D} k_2 \cosh k_2\ell$$

$$E = \frac{M_o}{D(k_1^2 + k_2^2)}\left(-k_1^2 \cos k_1\ell + \frac{I\omega^2}{D} k_1 \sin k_1\ell\right)$$

$$F = \frac{M_o}{D(k_1^2 + k_2^2)}\left(-k_2^2 \cosh k_2\ell + \frac{I\omega^2}{D} k_2 \sinh k_2\ell\right) \tag{A-29}$$

$$G = -k_1^3 \cos k_1\ell + \frac{m\omega^2}{D} \sin k_1\ell$$

$$H = k_2^3 \cosh k_2\ell + \frac{m\omega^2}{D} \sinh k_2\ell$$

$$J = \frac{M_o}{D(k_1^2 + k_2^2}\left(k_1^3 \sin k_1\ell + \frac{m\omega^2}{D} \cos k_1\ell\right)$$

and

$$L = \frac{M_o}{D(k_1^2 + k_2^2)} \left(- k_2^3 \sinh k_2 \ell - \frac{m\omega^2}{D} \cosh k_2 \ell \right).$$

Now the flexural impedance is given by

$$Z_m = \frac{M_o e^{-i\omega t}}{\left.\frac{\partial\phi}{\partial t}\right|_{t=o}}$$

with the angular velocity of the rib being

$$\left.\frac{\partial\phi}{\partial t}\right|_{t=o} = \frac{\partial}{\partial t}\left(\frac{\partial v(y,t)}{\partial y}\right)\bigg|_{y=o} . \tag{A-30}$$

Using the solution for v gives

$$\left.\frac{\partial\phi}{\partial t}\right|_{t=o} = -i\omega\ (k_1 C_0 + k_2 C_2) e^{-i\omega t}. \tag{A-31}$$

Thus

$$Z_{m\ \text{water-immersed}} = \frac{M_o e^{-i\omega t}}{-i\omega(k_1 C_0 + k_2 C_2) e^{-i\omega t}} = \frac{M_o}{-i\omega(k_1 C_0 + k_2 C_2)}$$

$$= \frac{AH-GB}{-i\omega\Big((E' + F')(-k_1 H + k_2 G) + (J' + L')(k_1 B - k_2 A)\Big)}, \tag{A-32}$$

where

$$E' = \frac{1}{D(k_1^2 + k_2^2)} \left(- k_1^2 \cos k_1 \ell + \frac{I\omega^2}{D} k_1 \sin k_1 \ell \right)$$

$$F' = \frac{1}{D(k_1^2 + k_2^2)} \left(- k_2^2 \cosh k_2 \ell + \frac{I\omega^2}{D} k_2 \sinh k_2 \ell \right)$$

$$J' = \frac{1}{D(k_1^2 + k_2^2)} \left(k_1^3 \sin k_1 \ell + \frac{m\omega^2}{D} \cos k_1 \ell \right) \tag{A-33}$$

$$L' = \frac{1}{D(k_1^2 + k_2^2)} \left(- k_2^3 \sinh k_2 \ell - \frac{m\omega^2}{D} \cosh k_2 \ell \right).$$